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W R Rossen

Department of Petroleum Engineering, University of Texas at Austin, Austin, TX 78712, USA

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Abstract. In bond percolation, for p just above p_c, there are clusters of dead-end open bonds that are each blocked from joining the backbone by a single closed bond. For any lattice, the *average* number of bonds in such clusters diverges as $(p-p_c)^{-1}$. We derive here formulae for the density of these clusters on any lattice, and for the size distribution and the density of clusters above any given size on the Bethe tree lattice. While the size of the *average* cluster diverges in magnitude as p approaches p_c, the density of clusters larger than any given finite size drops to zero at some $p > p_c$. In the mean-field approximation, for a fixed macroscopic voltage gradient, the voltage drop across the bond blocking such a cluster is proportional to the length of the cluster. The results presented here suggest that the maximum voltage drop across any blocked bond in an arbitrarily large, finite network does not approach infinity as p approaches p_c, as previously suggested, but approaches zero for some $p > p_c$. Analogous results are obtained for site percolation.

In the flow of current through sparse electrical networks with p, the open fraction, near the percolation threshold p_c, large voltage drops can develop across individual bonds. Examples are large voltage drops across open bonds for p just above p_c [1-4] and across closed bonds for p just below p_c [1, 2]. Recently [5] we addressed the problem of voltage drops across closed bonds for p just above p_c—that is, the voltage drop across a single closed bond that blocks a large cluster of dead-end open bonds from joining the conducting backbone. More precisely, we addressed the size of the clusters so blocked.

Our interest arises from the study of foams used to reduce gas flow through porous materials [6]. The liquid films, or lamellae, in these foams block the flow of gas as an insulating bond in a network blocks the flow of electricity. The crucial step in foam generation is the mobilization of these lamellae, which depends on the pressure drops across them exceeding the capillary resistance to their displacement. If there are too few lamellae to completely block gas flow, then the largest pressure drops occur across individual lamellae that block gas flow through long clusters of pores connected to the conducting backbone of gas-filled pores.

As a first, and rough, 'mean-field' approximation, we reasoned that this pressure drop across the lamella is proportional to the product of the size of the blocked cluster and the macroscopic pressure gradient ∇p. We then derived a formula for the average number of bonds in such clusters, $\bar{n}$ [5], and reasoned that the lamellae first mobilized were those which blocked clusters of some constant factor (e.g. 3) times the mean size. This led to an approximate formula for the macroscopic pressure gradient required to mobilize lamellae and generate foam. The average cluster size $\bar{n}$ approaches infinity as the fraction of bonds through which gas flows approaches the percolation threshold

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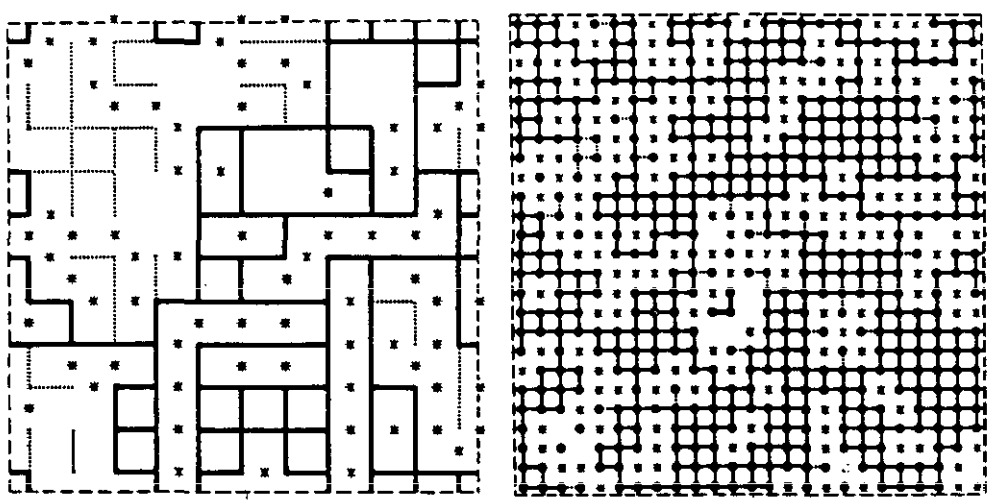
[5]; this suggests that the macroscopic pressure gradient required to mobilize the corresponding lamellae approaches zero.

In this paper, we derive new formulae for the number and distribution of sizes of these clusters, and revise our earlier conclusions. In the mean-field approximation, the pressure gradient required to mobilize lamellae does not approach zero as the percolation threshold is approached; ironically, it approaches infinity instead.

Our derivation focuses on bond percolation, but analogous arguments can be made for site percolation. For p near above the percolation threshold p_c, there are closed bonds that, if opened, would immediately join the backbone; we name these 'b^* bonds.' Examples are shown in figure 1. Each such bond blocks a 'dendritic,' 'tag end' or 'dead-end' cluster of open bonds that would join the backbone if the closed bond were opened; we call this cluster a 'b^* cluster' and define n to be the number of bonds in the cluster. (If only the b^* bond itself would join the backbone if it were opened, $n=1$.) A given dead-end open bond could belong to more than one b^* cluster, depending on the blocked paths by which it could rejoin the backbone, but a b^* bond defines exactly one b^* cluster.



(*a*) (*b*)

Figure 1. b^* Bonds on a square lattice. (*a*) bond percolation. 143 bonds out of 242 are occupied: backbone (——) 111 bonds; dendritic (· · · ·) 31 bonds; isolated (——) 1 bond. 99 bonds are unoccupied, of which 73 are b^* bonds (*). (*b*) Site percolation. 469 out of 625 sites are occupied: backbone (—●—) 431 sites; dendritic (· ·●· ·) 30 sites; isolated (—●—) 8 sites. 156 sites are unoccupied, of which 146 are b^* sites (*).

We define $\rho_b(n, p)$ to be the fraction of all bonds on the lattice that are b^* bonds blocking clusters of size n; in other words, $\rho_b(n, p)$ is the density of clusters of size n on the lattice. The average cluster size $\bar{n}$ is then

$$\bar{n} \equiv \sum_{n'=1}^{\infty} n' \rho_b(n', p) \left(\sum_{n'=1}^{\infty} \rho_b(n', p) \right)^{-1}. \tag{1}$$

For any lattice [5]

$$\bar{n} = p \, \mathrm{d}(\ln B)/\mathrm{d}p \tag{2}$$

where B is the backbone fraction; near the percolation threshold $\bar{n}$ scales as

$$\lim_{p \to p_c} \bar{n} = p_c \gamma_B (p - p_c)^{-1} \tag{3}$$

where γ_B is the exponent for the scaling of the backbone fraction near p_c [5].

Let ρ_{bt} be the fraction of all bonds on the network that are b^* bonds, or, in other words, the density of b^* bonds on the lattice:

$$\rho_{bt}(p) \equiv \sum_{n'=1}^{\infty} \rho_b(n', p). \tag{4}$$

One can derive a formula for ρ_{bt} as follows. Suppose the bonds in a lattice were being opened in a random sequence. The probability that the previously opened bond was a b^* bond is B/p, because (a) the previously opened bond is now on the backbone if and only if it had been a b^* bond before it was opened, and (b) any of the open bonds could have been the last opened with equal probability. Since these quantities do not change significantly upon opening a single bond on a large lattice, B/p is also the probability that the next bond opened will be a b^* bond; i.e., it is the probability that the next bond chosen at random from the closed fraction $(1-p)$ is a b^* bond. Therefore the total fraction ρ_{bt} of all bonds on the network that are b^* bonds is

$$\rho_{bt} = (1-p)B/p. \tag{5}$$

Equation (5) is valid for bond or site percolation on any lattice. Note that although the average b^* cluster, averaged over all b^* bonds, diverges in size as p approaches p_c according to (3), the density of b^* bonds on the network approaches zero as p approaches p_c, as governed by (5). Figure 2 shows ρ_{bt} for a Bethe tree lattice [7, 8] of coordination number $Z = 5$. For the Bethe tree, the percolation threshold $p_c = 1/(Z-1)$, or 0.25 for $Z = 5$. Thus, as figure 2 shows, there are no b^* bonds for $p < p_c = 0.25$ (because there is no backbone).

For the Bethe tree lattice, one can derive a formula for $\rho_b(n, p)$, as follows. This derivation is summarized schematically in figure 3. The bond of interest is the one

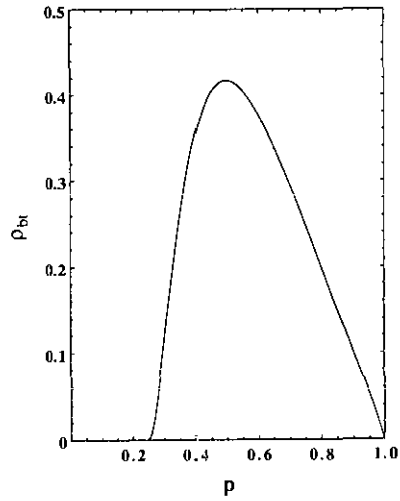


Figure 2. Density of b^* bonds on Bethe tree lattice, $Z = 5$.

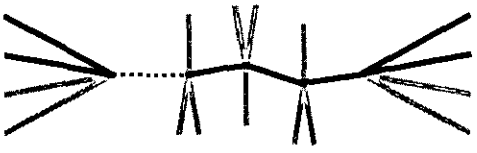


Figure 3. Derivation of $\rho_b(n, p)$.

second from the left in the figure. The probability that this bond is a b^* bond is the product of five factors.

(a) The bond itself must be closed: probability $(1-p)$.

(b) We assume first that the b^* bond is the leftmost bond of the b^* cluster. Therefore, the blocked bond must touch the backbone on its left. In other words, of the $(Z-1)$ bonds to the left, at least two of them must be part of an infinite cluster extending away from the blocked bond. The factor R defined for the Bethe tree by Stinchcombe [7] is the probability that a given bond is *not* part of such a cluster. It is the root of the equation [8]

$$\left[\sum_{j=2}^{Z-1} R^{(z-j)}\right] + (p-1)/p = 0 \tag{6}$$

that goes to zero as p approaches 1. The probability that the backbone touches the left end of the bond of interest is one minus the probability that none of the $(Z-1)$ bonds are part of an infinite cluster extending to the left, minus the probability that exactly one of the bonds is part of such a cluster:

$$[1 - R^{(Z-1)} - (1-R)R^{(Z-2)}(Z-1)].$$

(c) The next bond in the b^* cluster must be open, and none of the other $(Z-2)$ bonds may belong to infinite clisters extending away from the b^* cluster. (Otherwise the backbone would pass through this point.) The b^* cluster could extend through any of the $(Z-1)$ bonds of this group with equal probability. This event must continue through the $(n-1)$ open bonds of the b^* cluster:

$$[pR^{(Z-2)}(Z-1)]^{(n-1)}.$$

(d) At the right of the b^* cluster, the situation is the same as at the left end (b): at least two of the $(Z-1)$ bonds must be part of infinite open clusters that do not include the b^* cluster:

$$[1 - R^{(Z-1)} - (1-R)R^{(Z-2)}(Z-1)].$$

(e) Initially at (b) we assumed that the closed bond was the leftmost of the b^* cluster. It could be any of the n bonds in the cluster with equal probability; therefore the overall probability is n times that computed with the other four factors.

The final result is

$$\rho_b(n, p) = n(1-p)[1 - R^{(Z-1)} - (1-R)R^{(Z-2)}(Z-1)]^2[pR^{(Z-2)}(Z-1)]^{(n-1)}. \tag{7}$$

For fixed p, $\rho_b(n, p)$ has the form

$$\rho_b(n, p) = nu_1(p)u_2(p)^{(n-1)}. \tag{8}$$

Similar arguments lead to the corresponding formula for site percolation:

$$\rho_s(n, p) = n(1-p)[1-R^{(Z-1)}-(1-R)^{(Z-2)}(Z-1)]^2 p[pR^{(Z-2)}(Z-1)]^n \tag{9}$$

which follows scaling similar to (8). The factor $R(p)$ is the same for site or bond percolation [7, 8].

Figure 4 shows $\rho_b(n, p)$ for a Bethe tree with $Z = 5$ for several values of p. From (2), the values of $\bar{n}$ are 49.9, 16.6, 9.9, 4.9 and 3.2 for $p = 0.26$, 0.28, 0.3, 0.35 and 0.4. The values obtained by direct summation using (7) for n up to 300 agree with these values; slightly different values are obtained from the asymptotic formula (3). The values of ρ_{bt} obtained from either summing (7) over n or from (5) are 0.007 452, 0.051 87, 0.1129, 0.2599 and 0.3571 for the same values of p.

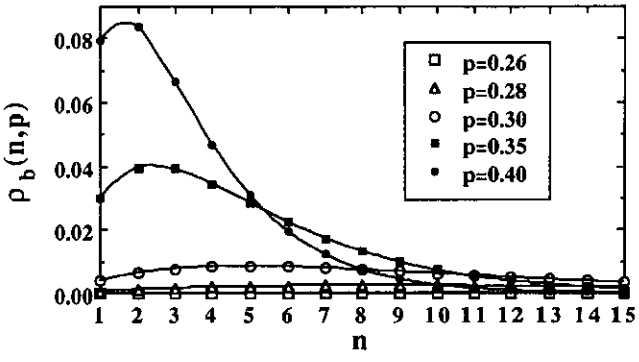


Figure 4. Density of blocked clusters on Bethe tree lattice, $Z = 5$.

Using (7) or (9) one can estimate the fraction of bonds that are b^* bonds blocking clusters above some certain threshold size; or equivalently, the size such that a given fraction of bonds are b^* bonds that block clusters of that size or greater. Let $n_b(\varepsilon_b, p)$ be the size such that a fraction ε_b of the bonds in the network are b^* bonds blocking clusters of size n_b or greater. In other words, $n_b(\varepsilon_b, p)$ satisfies the equation

$$\sum_{n'=n_b}^{\infty} \rho_b(n', p) = \varepsilon_b. \tag{10}$$

Substituting an integral for a summation in (10) is valid for the large values of n' involved. Using (8), (10) becomes

$$u_1[u_2^{(n_b-1)}/(\ln u_2)^2][1-(\ln u_2)n_b] = \varepsilon_b \tag{11}$$

with

$$u_1 \equiv (1-p)[1-R^{(Z-1)}-(1-R)R^{(Z-2)}(Z-1)]^2$$

$$u_2 \equiv pR^{(Z-2)}(Z-1).$$

For site percolation, the corresponding formula is

$$u_1[u_2^{n_b}/(\ln u_2)^2][1-(\ln u_2)n_b] = \varepsilon_s \tag{12}$$

with

$$u_1 \equiv p(1-p)[1-R^{(Z-1)}-(1-R)^{(Z-2)}(Z-1)]^2$$

$$u_2 \equiv pR^{(Z-2)}(Z-1).$$

Figure 5 shows $n_b(\varepsilon_b, p)$ for several values of ε_b for the Bethe tree with $Z = 5$. Also shown is $\bar{n}$ and $3\bar{n}$. Far from the percolation threshold, a given multiple of $\bar{n}$ is approximately equivalent to $n_b(\varepsilon_b, p)$ for some fixed value of ε_b. The value $3\bar{n}$, which we used in a mean-field model for pressure drops across liquid films in porous media [6], corresponds to a fraction of between 0.001 and 0.01 of the bonds on the lattice. In other words, roughly 1 bond in 300 is a b^* bond blocking a cluster of size equal to or greater than $3\bar{n}$, over a range of values of p far from the percolation threshold. Sufficiently near the percolation threshold, however, $n_b(\varepsilon_b, p)$ is zero for any finite value of ε_b, because the fraction of all b^* bonds approaches zero at the percolation threshold (5). In other words, any finite multiple of $\bar{n}$ overestimates $n_b(\varepsilon_b, p)$ for any fixed values of ε_b near the percolation threshold. In our mean-field model for flow in porous media, this implies that the number of lamellae in a network of finite size that can be mobilized at any finite pressure gradient approaches zero as $p \to p_c$; the average cluster size approaches infinity, but the number of clusters approaches zero. Hence the minimum macroscopic pressure gradient for foam generation approaches infinity, not zero, as $p \to p_c$.

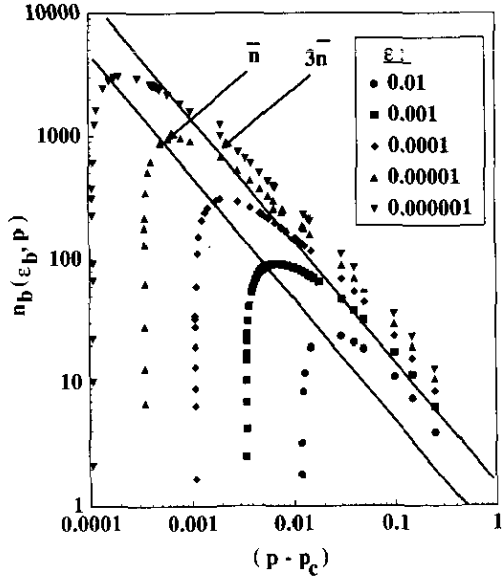


Figure 5. Cluster size near percolation threshold: Bethe tree; $Z = 5$.

Near the percolation threshold, true potential (voltage or pressure) drops would deviate from the mean-field model due to the inhomogeneity of the current distribution [1-4], but no general, quantitative model for potential drops across blocked bonds under these conditions is yet available.

Acknowledgments

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References

[1] Duxbury P M, Beale P D and Leath P L 1986 *Phys. Rev. Lett.* **57** 1052
[2] de Arcangelis L, Redner S and Coniglio A 1986 *Phys. Rev.* B **36** 4656
[3] Duxbury P M, Leath P L and Beale P D 1987 *Phys. Rev.* B **36** 367
[4] Kahng B, Batrouni G G and Redner S 1987 *J. Phys. A: Math. Gen.* **20** L827
[5] Rossen W R 1988 *J. Phys. A: Math. Gen.* **21** L533
[6] Rossen W R and Gauglitz P A 1990 *AIChE J.* **36** 1176
[7] Stinchcombe R B 1974 *J. Phys. C: Solid State Phys.* 7 179
[8] Larson R G and Davis H T 1982 *J. Phys. C: Solid State Phys.* **15** 2327